Subfactor Theory and Quantum Groupoids

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Abstract

The purpose of this presentation is to give an overview of the connection between Jones' Subfactor Theory and Operator Algebraic Quantum Groupoids. That is why the present work gives a quick survey of the original motivation behind this connection, the relevant results obtained so far, and some ongoing research projects.

Motivation and relevant results

Let G be a finite group acting outerly on a II_1-factor M (for example on the hyperfinite II_1-factor $\mathcal{R}$). Then, the inclusion $M^G \subset M$ yields an irreducible depth 2 II_1-subfactor inclusion of finite index $|G|$. Applying the Jones' basic construction, we obtain the following tower of II_1-factors

$$M^G \subset M \subset M \rtimes G.$$

A finite group G is a particular example of a finite-dimensional Kac algebra (a finite quantum group). Kac algebras was originally introduced in order to study a general Pontrjagin duality in the case of non abelian locally compact groups.

Similar to the case of a finite group, given a finite-dimensional Kac algebra $\mathbb{K}$ acting outerly on a II_1-factor M (for example on the hyperfinite II_1-factor $\mathcal{R}$), we obtain an irreducible depth 2 II_1-subfactor inclusion $M^{\mathbb{K}} \subset M$ with finite index and with basic construction given by

$$M^{\mathbb{K}} \subset M \subset M \rtimes \mathbb{K}.$$

It was announced by A. Ocneanu [1] that irreducible depth 2 II_1-subfactor inclusions of finite index can be characterized in terms of crossed products of finite-dimensional Kac algebras. This conjecture was achieved with the following theorem:

Ocneanu's theorem (Szymański '94, Longo '94, David '96)

Let $N \subset M$ be an irreducible depth 2 II_1-subfactor inclusion of finite index. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two finite-dimensional Kac algebra structures on $M' \cap M_3$ and $N' \cap M_2$, dual each other, denoted by $\mathbb{K}$ and $\hat{\mathbb{K}}$ respectively, an outer action of $\mathbb{K}$ on M and an outer action of $\hat{\mathbb{K}}$ on N such that

$$N = M^{\mathbb{K}}, \qquad M_2 \cong M \rtimes \mathbb{K}, \quad \text{and} \quad M \cong N \rtimes \hat{\mathbb{K}}.$$

It seems natural to try to find out if Ocneanu's theorem can be generalize to a more general case, for example if:

- $N \subset M$ is an irreducible depth 2 subfactor inclusion with no restriction on the value of the index;
- $N \subset M$ is a depth 2 II_1-subfactor inclusion of finite index.

We now know that similar results follow in these two cases. And, in order to explain one of the main motivations for the study of operator algebraic quantum groupoids, we give in the following lines a description of these two generalizations of Ocneanu's theorem.

Inclusions and compact/discrete quantum groups

In [2], the authors gave the first steps of a possible generalization for the case of an irreducible depth 2 subfactor inclusion of index not necessarily finite. In their work, they characterize the inclusion of semi-finite factors using crossed product by twisted actions of discrete groups, and they conjecture a result in the case of discrete Kac algebras.

In [6], the conjecture was finally proved using for it the general framework of operator algebraic quantum groups (mainly the theory of multiplicative unitaries in the sense of Baaj-Skandalis).

Theorem (Enock-Nest '96 + Enock '98)

Let $N \subset M$ be an irreducible depth 2 subfactor inclusion, equipped with a normal semi-finite faithful operator-valued weight T from M to N satisfying some regular condition. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two locally compact quantum group structures on $M' \cap M_3$ and $N' \cap M_2$, dual to each other, denoted by $\mathbb{G}$ and $\hat{\mathbb{G}}$ respectively; an outer action of $\mathbb{G}$ on M and an outer action of $\hat{\mathbb{G}}$ on N such that

$$N = M^{\mathbb{G}}, \qquad M_2 \cong M \rtimes \mathbb{G}, \quad \text{and} \quad M \cong N \rtimes \hat{\mathbb{G}}.$$

Remark: Any locally compact quantum group arises in that way ([16]).

As corollary, it was shown that:

- If the inclusion $N \subset M$ is compact, the quantum group $\mathbb{G}$ is a compact Kac algebra.
- If the inclusion $N \subset M$ is discrete, the quantum group $\mathbb{G}$ is a discrete Kac algebra.
- If the inclusion $N \subset M$ is compact and discrete (equivalently $N \subset M$ is of finite index), the quantum group $\mathbb{G}$ is a finite-dimensional Kac algebra.

Remark: In [9], a Galois correspondence was shown for action of compact groups on von Neumann algebras.

Inclusions and finite quantum groupoids

It was suggested in [7] the possibility to characterize finite index depth 2 II_1-subfactor inclusions in terms of finite-dimensional weak Hopf C*-algebras. Weak Hopf C*- algebras and Weak Kac algebras was introduced previously as a generalization of Kac algebras and groupoids algebras.

Similar to Kac algebras, given a finite-dimensional weak Kac algebra $\mathbb{K}$ acting outerly on a II_1-factor M (for example on the hyperfinite II_1-factor $\mathcal{R}$), we obtain a depth 2 II_1-subfactor inclusion $M^{\mathbb{K}} \subset M$ with finite index such that its basic construction is given by

$$M^{\mathbb{K}} \subset M \subset M \rtimes \mathbb{K}.$$

In this case, $\mathbb{K}$ acting outerly on M means that $(M^{\mathbb{K}})' \cap M = C(\mathbb{K})_s$, where $C(\mathbb{K})_s$ denotes the source counit subalgebra of $\mathbb{K}$, then the inclusion above $M^{\mathbb{K}} \subset M$ is not necessarily irreducible since the source counit subalgebra $C(\mathbb{K})_s$ for a weak Kac algebra $\mathbb{K}$ is not necessarily a trivial C*-subalgebra. In fact, $C(\mathbb{K})_s = \mathbb{C}$ if and only if $\mathbb{K}$ is a Kac algebra.

In [11], Ocneanu's theorem has been extended to the framework of finite quantum groupoids (weak Hopf C*-algebras and weak Kac algebras). Moreover, in [14], a Galois correspondence was shown for finite depth II_1-subfactor inclusions of finite index. This correspondence makes it possible to share information between finite quantum groupoids and II_1-subfactor inclusions of finite index, for example concerning the categorical data associated with these objects.

Theorem (Nikshych-Vainerman '00 + Nikshych-Vainerman '00)

Let $N \subset M$ be a depth 2 II_1-subfactor inclusion of finite index. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two finite-dimensional weak Hopf C*-algebra structures on $M' \cap M_3$ and $N' \cap M_2$, dual each other, denoted by $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ respectively, an outer action of $\mathfrak{G}$ on M and an outer action of $\hat{\mathfrak{G}}$ on N such that

$$N = M^{\mathfrak{G}}, \qquad M_2 \cong M \rtimes \mathfrak{G}, \qquad M \cong N \rtimes \hat{\mathfrak{G}},$$

and $[M:N] = \dim(\mathfrak{G}) := \|\Lambda_{C(\mathfrak{G})_s}^{C(\mathfrak{G})}\|^2$. Moreover, we have the equivalences of categories

$${}_N\mathrm{Bim}_N(N \subset M) \cong \mathrm{Rep}(\mathfrak{G}) \quad \text{and} \quad {}_M\mathrm{Bim}_M(M \subset M_2) \cong \mathrm{Rep}(\hat{\mathfrak{G}}).$$

Operator algebraic quantum groupoids

A more general question arises from the two generalizations above: Is it possible to give a similar result in the general case of inclusions of von Neumann algebras?

In [10, 13, 15], a positive answer to the last question can be found, under certain technical conditions on the inclusion. Moreover, it was necessary to introduce a new quantum object which extends the definition of a locally compact quantum group. This led to the current definition of a measured quantum groupoid.

So far, a general statement of Ocneanu's theorem is:

Theorem (Enock-Vallin '00 + Enock '00 + Enock '05)

Let $N \subset M$ be an inclusion of σ-finite von Neumann algebras of depth 2, equipped with a regular normal semi-finite faithful operator-valued weight T from M to N. Suppose there exists on $N' \cap M$ an adapted faithful semi-finite weight μ and consider the associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are a measured quantum groupoid structure on $M' \cap M_3$, denoted by $\mathfrak{G} = \mathfrak{G}(N \subset M)$, and an outer action of $\mathfrak{G}$ on M such that

$$N \cong M^{\mathfrak{G}}, \qquad M_2 \cong M \rtimes \mathfrak{G}.$$

Moreover, there are a measured quantum groupoid structure on $N' \cap M_2$, denoted by $\hat{\mathfrak{G}}$, which is the Pontrjagin dual of $\mathfrak{G}$, and an outer action of $\hat{\mathfrak{G}}$ on N such that $M \cong N \rtimes \hat{\mathfrak{G}}$. Using these measured quantum groupoids, the Jones' tower $(M_i)_{i\in\mathbb{N}}$ is given by

$$M^{\mathfrak{G}} \subset M \subset M \rtimes \mathfrak{G} \subset (M \rtimes \mathfrak{G}) \rtimes \hat{\mathfrak{G}} \subset \cdots$$

Remark: Any measured quantum groupoid arises in that way ([17]).

Ongoing research projects

Nowadays, the theory of measured quantum groupoids seems to be the correct theory of quantum groupoids in the framework of operator algebras, but this theory is still in an early stage. Some examples of measured quantum groupoids which are neither quantum groups nor finite quantum groupoids have been given but the measured quantum transformation groupoids introduced by M. Enock and T. Timmermann [18] are the most interesting. These quantum groupoids are constructed using special kind of actions of locally compact quantum groups on von Neumann algebras. So far, it seems that these quantum objects play an important role in the theory of operator algebraic quantum groupoids and then it is natural to study in more detail its structure and its connection with inclusions of von Neumann algebras.

Inclusions and compact/discrete quantum transformation groupoids

In [19, 20], using measured Yetter–Drinfeld C*-algebras over compact quantum groups, we introduce a class of compact quantum groupoids called compact quantum transformation groupoids. These quantum groupoids are examples of measured quantum transformation groupoids of compact type. Explicitly, given a unital measured Yetter–Drinfeld C*-algebra $(N, \theta, \hat{\theta}, \mu)$ over a compact quantum group $\mathbb{G}$, there exists a compact quantum groupoid structure on $\mathbb{G} \ltimes N$, denoted by $\mathfrak{G}(N, \theta, \hat{\theta}, \mu)$ and called a compact quantum transformation groupoid. Moreover, the Pontrjagin dual of $\mathfrak{G}(N, \theta, \hat{\theta}, \mu)$ is given by the discrete quantum groupoid $\mathfrak{G}(N^{\mathrm{op}}, \hat{\theta}, \theta, \mu)$, called a discrete quantum transformation groupoid. Among the main examples of such compact quantum transformation groupoids, we have

- Any compact transformation groupoid yields a commutative compact quantum transformation groupoid.
- Finite-dimensional measured Yetter–Drinfeld C*-algebras over a finite-dimensional Kac algebras yields compact quantum transformation groupoids which are finite-dimensional weak Kac algebras.
- Any compact quantum group is a compact quantum transformation groupoid.

Open questions: Using the connection between inclusions of von Neumann algebras of depth 2 and measured quantum groupoids, it seems natural to try to answer the following open questions:

- Similar to the case of compact/discrete Kac algebras. What kind of inclusions can be found related to compact/discrete quantum transformation groupoids?
- Similar to the case of finite quantum groupoids. Is it possible to give a connection between some categorical data associated to inclusions of von Neumann algebras and compact quantum transformation groupoids?
- Is there a Galois correspondence for actions of compact quantum transformation groupoids on von Neumann algebras that generalizes the known results for compact groups and finite quantum groupoids?

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